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Current Fluctuations of the One Dimensional Symmetric Simple Exclusion Process with a Step Initial Condition

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For the symmetric simple exclusion process on an infinite line, we calculate exactly the fluctuations of the integrated current Q_t during time t through the origin when, in the initial condition, the sites are occupied with density ρ_a on the negative axis and with density ρ_b on the positive axis. All the cumulants of Q_t grow like \sqrt{t} . In the range where $Q_t \sim \sqrt{t}$, the decay $\exp[-Q_t^3/t]$ of the distribution of Q_t is non-Gaussian. Our results are obtained using the Bethe ansatz and several identities recently derived by Tracy and Widom for exclusion processes on the infinite line.

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I. INTRODUCTION

Understanding how currents of particles or of heat fluctuate through non equilibrium systems has motivated lots of efforts over the last two decades. A number of general symmetries of the distribution of these fluctuations, based on the microreversibility of the dynamics, have been discovered [11, 20, 21, 29]. Beyond these symmetries, the distribution of the current fluctuations has been calculated in several cases [9, 22, 25, 26, 27, 36]. For diffusive systems, a general theory, the macroscopic fluctuation theory, has been developed [6] which, under some conditions [7, 8, 10], allows one to calculate the whole distribution of these current fluctuations for systems maintained in a non-equilibrium steady state by contact with two reservoirs (of particles at different chemical potentials or of heat at different temperatures) [9, 17, 27]. For driven systems, which are not diffusive, the fluctuations belong to the universality class of the Kardar Parisi Zhang equation [31] and can be related to the fluctuations in growth models and to the theory of random matrices [30, 35, 36, 40, 42, 48, 49].

For diffusive systems as well as for driven systems, it has been very helpful to analyse simple models, in particular exclusion processes [32, 34, 47]. In exclusion processes, one considers particles hopping on a lattice with a hard core interaction which prevents two particles from being on the same lattice site. In one dimension, exclusion processes can be solved by the Bethe ansatz [4, 12, 13, 15, 23, 24, 39, 48, 49, 50] and a number of exact results have been obtained on the fluctuation of the current or the distribution of particles for a given initial condition [19, 37, 38, 44].

In the present work we consider the symmetric simple exclusion (SSEP) on an infinite one dimensional lattice. By definition of the model each lattice site is either empty or occupied by a single particle. The dynamics is stochastic: each particle hops to each of its neighboring sites at rate 1 if the move is not forbidden by the exclusion rule (each site is occupied by at most one particle). At time $t = 0$ each site at the left of the origin ($x \leq 0$) is occupied with a probability ρ_a and each site at the right of the origin ($x > 0$) is occupied with a probability ρ_b (see figure 1). We also assume that the measure at $t = 0$ is Bernoulli, meaning that there are no correlation between the occupation numbers of the different sites in the initial condition. We call Q_t the total flux of particles between site 0 and site 1 during the time interval t . Our goal in the present paper is to determine, for large times t , the probability distribution of Q_t .

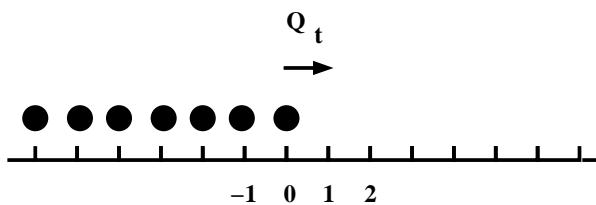


FIG. 1: The initial condition when $\rho_a = 1$ and $\rho_b = 0$

The same initial condition has already been considered for the asymmetric simple exclusion (ASEP) [3, 28, 36, 42, 44, 48, 49, 50], in particular in connection to random matrix theory.

Our main result is that, for large t , the generating function of this total flux is given by

$$\langle e^{\lambda Q_t} \rangle \sim e^{\sqrt{t}F(\omega)} \quad (1)$$

where the function $F(\omega)$ is defined by

$$F(\omega) = \frac{1}{\sqrt{\pi}} \sum_{n \geq 1} \frac{(-)^{n+1}}{n^{3/2}} \omega^n \equiv \frac{1}{\pi} \int_{\infty}^{\infty} dk \log [1 + \omega e^{-k^2}] \quad (2)$$

and ω is a function of ρ_a, ρ_b, λ

$$\omega = \rho_a(e^{\lambda} - 1) + \rho_b(e^{-\lambda} - 1) + \rho_a \rho_b (e^{\lambda} - 1)(e^{-\lambda} - 1). \quad (3)$$

Remark: It has already been noticed [14] that in the steady state of the SSEP on a large but finite chain, the generating function of the current was also a function of ρ_a, ρ_b, λ through the same single parameter, ω . This is a general property of the symmetric exclusion process that we shall establish in section 2 for a general graph.

Remark: From (1,2,3) one can obtain all the cumulants of Q_t , in the large t limit. They all grow like \sqrt{t} and the prefactor in the cumulant $\langle Q_t^n \rangle_c$ is a polynomial of degree n in ρ_a and ρ_b . For example, for $\rho_a = \rho$ and $\rho_b = 0$, one gets

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{-1/2} \langle Q_t \rangle_c &= \frac{1}{\sqrt{\pi}} \rho, \\ \lim_{t \rightarrow \infty} t^{-1/2} \langle Q_t^2 \rangle_c &= \frac{1}{\sqrt{\pi}} \left(\rho - \frac{\rho^2}{\sqrt{2}} \right), \\ \lim_{t \rightarrow \infty} t^{-1/2} \langle Q_t^3 \rangle_c &= \frac{1}{\sqrt{\pi}} \left(\rho - \frac{3}{\sqrt{2}} \rho^2 + \frac{2}{\sqrt{3}} \rho^3 \right), \end{aligned}$$

while for $\rho_a = \rho_b = \rho$ one gets for the even cumulants (all the odd ones vanish by symmetry)

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{-1/2} \langle Q_t^2 \rangle_c &= \frac{2}{\sqrt{\pi}} \rho(1 - \rho), \\ \lim_{t \rightarrow \infty} t^{-1/2} \langle Q_t^4 \rangle_c &= \frac{2}{\sqrt{\pi}} \rho(1 - \rho) - \frac{6\sqrt{2}}{\sqrt{\pi}} \rho^2(1 - \rho)^2. \end{aligned}$$

This can be compared to the variance of the position X_t of a tagged particle [2, 5, 33, 41, 43, 51], which also grows like \sqrt{t} , by arguing that the typical distance between particles is ρ^{-1} so that $\langle Q_t^2 \rangle \simeq \rho^2 \langle X_t^2 \rangle$.

From the knowledge of the generating function (1), one can obtain the distribution of Q_t . This distribution takes, in the long time limit, a scaling form

$$\text{Pro} \left(\frac{Q_t}{\sqrt{t}} = q \right) \sim \exp[\sqrt{t}G(q)], \quad (4)$$

where $G(q)$ can be related to $F(\omega)$ by a Lagrange transform. This allows one to obtain the asymptotics of $G(q)$. One can see from the integral representation (2) of $F(\omega)$ that, for large positive ω ,

$$F(\omega) \simeq \frac{4}{3\pi} [\log \omega]^{3/2} + \frac{\pi}{6} [\log \omega]^{-1/2} + ..$$

and this gives for large positive q

$$G(q) \simeq -\frac{\pi^2}{12} q^3 + q \log(\rho_a(1 - \rho_b)) + .. \quad (5)$$

This paper is organized as follows: in section II we prove that the generating function $\langle e^{\lambda Q_t} \rangle$ of Q_t is a function of the densities ρ_a, ρ_b and of λ through the single parameter ω defined in (3). In section III we obtain an exact expression of the generating function of Q_t from which we derive the large t asymptotics (1,2).

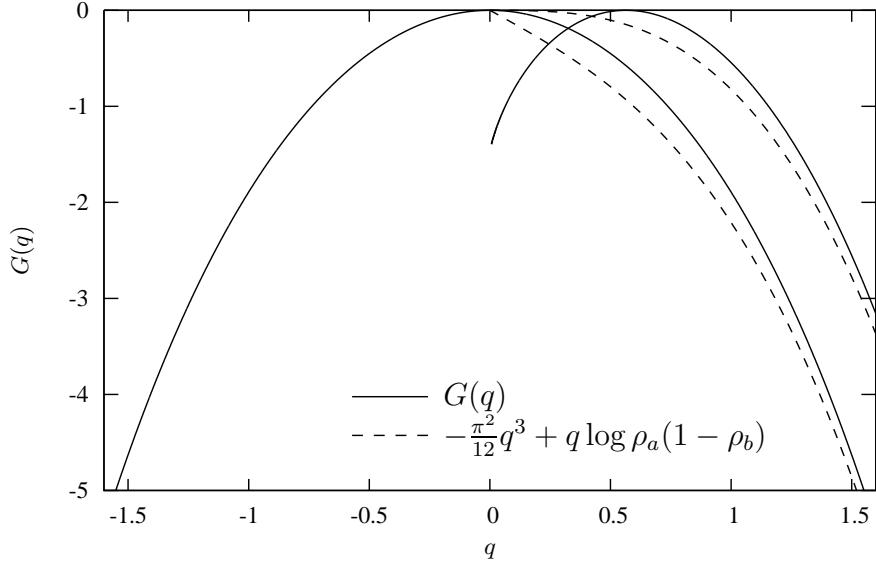


FIG. 2: The large deviation function $G(q)$ and its asymptotics for large q (dashed lines) in the cases $\rho_a = \rho_b = 1/2$ (left curves) and $\rho_a = 1$, $\rho_b = 0$ (right curves).

II. EVOLUTION OF THE CORRELATION FUNCTIONS AND THE ω DEPENDENCE

Let τ_x be a binary variable which indicates whether site x is occupied ($\tau_x = 1$) or empty ($\tau_x = 0$). In this section we are going to argue that all the n -point correlation functions of these τ_x 's are polynomials of degree n in ρ_a and ρ_b . This will allow us to show that the n -th moment $\langle Q_t^n \rangle$ is also a polynomial of degree n in ρ_a and ρ_b . Then using the particle-hole symmetry, we will show that the generating function $\langle e^{\lambda Q_t} \rangle$ depends on ρ_a , ρ_b and λ through the single parameter ω defined in (3).

One can write the exact evolution of the expectation $\langle \tau_x \rangle$ where $\langle \cdot \rangle$ denotes an average over the stochastic evolution and the initial conditions. On the infinite line it takes the form

$$\frac{d}{dt} \langle \tau_x \rangle = \langle \tau_{x+1} \rangle + \langle \tau_{x-1} \rangle - 2\langle \tau_x \rangle. \quad (6)$$

As at time $t = 0$, one has $\langle \tau_x \rangle = \rho_a$ for $x \leq 0$ and $\langle \tau_x \rangle = \rho_b$ for $x > 0$, it is clear that at any later time the solution of (6) is linear in ρ_a and ρ_b .

Similarly for any $x < y$ the evolution of the 2-point correlation function is given for $y > x + 1$ by

$$\frac{d}{dt} \langle \tau_x \tau_y \rangle = \langle \tau_{x+1} \tau_y \rangle + \langle \tau_{x-1} \tau_y \rangle + \langle \tau_x \tau_{y+1} \rangle + \langle \tau_x \tau_{y-1} \rangle - 4\langle \tau_x \tau_y \rangle \quad (7)$$

and for $y = x + 1$ by

$$\frac{d}{dt} \langle \tau_x \tau_{x+1} \rangle = \langle \tau_{x-1} \tau_{x+1} \rangle + \langle \tau_x \tau_{x+2} \rangle - 2\langle \tau_x \tau_{x+1} \rangle. \quad (8)$$

These evolution equations do not involve higher correlation functions and as at $t = 0$ the only possible values of $\langle \tau_x \tau_y \rangle$ are ρ_a^2 , $\rho_a \rho_b$ or ρ_b^2 , one can easily see that at any later time $\langle \tau_x \tau_y \rangle$ remains a quadratic function of ρ_a and ρ_b .

One can generalize this property of the SSEP to all equal-time or unequal-time correlation functions on a general graph. Let us consider the symmetric exclusion process on such a graph : the vertices are either empty or occupied by a single particle and each particle can hop at rate one to every empty site to which it is directly connected by the graph. Therefore if $X \equiv \{x_1, \dots, x_n\}$ is a set of n different sites on the graph and if $\langle \tau_X \rangle$ denotes their n -point correlation function

$$\langle \tau_X \rangle = \langle \tau_{x_1} \dots \tau_{x_n} \rangle,$$

it is easy to see that the evolution of $\langle \tau_X \rangle$ is given by

$$\frac{d\langle \tau_X \rangle}{dt} = \sum_{y \notin X} \sum_{k=1}^n \chi(y, x_k) [\langle \tau_y \tau_{X \setminus \{x_k\}} \rangle - \langle \tau_X \rangle] \quad (9)$$

where $\chi(x, y) = 1$ or 0 indicates whether the edge (x, y) is present or not on the graph. If initially all the τ_x 's are uncorrelated, some sites being occupied with probability ρ_a and all the other sites occupied with probability ρ_b , all the equal-time n -point correlation functions are obviously polynomials of degree exactly n in ρ_a and ρ_b at time $t = 0$. This property is preserved by the evolution (9). A similar reasoning allows one to show that all the unequal time n -point correlation functions are also polynomials in ρ_a and ρ_b of degree at most n .

On this general graph let us call A the subset of sites occupied at time $t = 0$ with probability ρ_a (all the other sites being occupied with probability ρ_b). Obviously one has $\langle \tau_x \rangle = \rho_a$ if $x \in A$ and $\langle \tau_x \rangle = \rho_b$ if $x \notin A$. If Q_t is the total flux out of the subset A during time t , one has

$$Q_t = \sum_{x \in A} \tau_x(0) - \tau_x(t). \quad (10)$$

It is clear from (10) that $\langle Q_t^n \rangle$ can be expressed in terms of unequal-time n -point correlation functions and is therefore a polynomial of degree n in ρ_a and ρ_b (if one or more sites are repeated more than once in the correlation function, this may give a polynomial of lower degree as $\tau_x^k = \tau_x$ for all $k \geq 1$).

To show that $\langle e^{\lambda Q_t} \rangle$ depends only on the reduced parameter ω , let us consider a finite lattice of N sites, where initially all the N_A sites of the subset A are occupied with density ρ_a , and all the remaining $N - N_A$ sites are occupied with density ρ_b . One can write the generating function $\langle e^{\lambda Q_t} \rangle$ as

$$\langle e^{\lambda Q_t} \rangle = \sum_{p=0}^{N_A} \sum_{q=0}^{N-N_A} \rho_a^p (1 - \rho_a)^{N_A-p} \rho_b^q (1 - \rho_b)^{N-N_A-q} e^{-q\lambda} R_{p,q}(e^\lambda), \quad (11)$$

where $R_{p,q}(z)$ is a polynomial of degree $p + q$ in z which depends on time and on the choice of the subset A . The reason for the factor $e^{-q\lambda} R_{p,q}(e^\lambda)$ is that this term corresponds to situations where initially there are p particles in the subgraph A and q particles in the rest of the graph so that Q_t can only take the values $Q_t = -q, -q+1, \dots, p$. If one expands this expression in powers of ρ_a and ρ_b , one gets

$$\langle e^{\lambda Q_t} \rangle = \sum_{p=0}^{N_A} \sum_{q=0}^{N-N_A} \rho_a^p \rho_b^q e^{-q\lambda} S_{p,q}(e^\lambda), \quad (12)$$

where $S_{p,q}(z)$ is also a polynomial of degree $p + q$. For $\langle Q_t^n \rangle$ to be a polynomial of degree n in ρ_a and ρ_b one needs that for λ small

$$S_{p,q}(e^\lambda) = O(\lambda^{p+q}),$$

so that the polynomial $S_{p,q}$ has the form

$$S_{p,q}(e^\lambda) = s_{p,q}(e^\lambda - 1)^{p+q},$$

where $s_{p,q}$ is a number which depends on time, on the graph and on the subgraph A but does not depend on λ . This implies that (11) can be rewritten as

$$\langle e^{\lambda Q_t} \rangle = \sum_{p=0}^{N_A} \sum_{q=0}^{N-N_A} s_{p,q} [\rho_a(e^\lambda - 1)]^p [\rho_b(e^{-\lambda} - 1)]^q \equiv G(\rho_a(e^\lambda - 1), \rho_b(e^{-\lambda} - 1)). \quad (13)$$

This shows clearly that $\langle e^{\lambda Q_t} \rangle$ is already a function of only two reduced variables: $\rho_a(e^\lambda - 1)$ and $\rho_b(e^{-\lambda} - 1)$.

Let us now use the particle-hole symmetry. In the SSEP, holes have exactly the same dynamics as particles. Therefore the generating function of the flux Q_t is left invariant by the particle-hole symmetry $(\rho_a, \rho_b, \lambda) \rightarrow (1 - \rho_a, 1 - \rho_b, -\lambda)$. In terms of the function G of two variables defined in (13), this means that, for any ρ_a, ρ_b, λ

$$G(\rho_a(e^\lambda - 1), \rho_b(e^{-\lambda} - 1)) = G((1 - \rho_a)(e^{-\lambda} - 1), (1 - \rho_b)(e^\lambda - 1)), \quad (14)$$

and so, if $\alpha = \rho_a(e^\lambda - 1)$ and $\beta = \rho_b(e^{-\lambda} - 1)$, one has

$$G(\alpha, \beta) = G(e^{-\lambda} - 1 + \alpha e^{-\lambda}, e^\lambda - 1 + \beta e^\lambda). \quad (15)$$

As (15) is valid for any λ , one can choose $e^{-\lambda} = 1 + \beta$, which leads to

$$G(\alpha, \beta) = G(\alpha + \beta + \alpha\beta, 0). \quad (16)$$

This completes the proof that $G(\alpha, \beta)$ and therefore $\langle e^{\lambda Q_t} \rangle$ are functions of the single variable $\alpha + \beta + \alpha\beta = \rho_a(e^\lambda - 1) + \rho_b(e^{-\lambda} - 1) + \rho_a(e^\lambda - 1)\rho_b(e^{-\lambda} - 1) \equiv \omega$.

Remark: This ω dependence of $\langle e^{\lambda Q_t} \rangle$ was already noticed for the SSEP on a finite lattice of length L connected at its extremities to two reservoirs of particles in [14], where it was, however, only obtained in the large t and L limit. A consequence of the above discussion is that the ω dependence remains valid for any system size L , at any time t , in more complicated geometries (in particular in higher dimensions), and for more complicated connections to the two reservoirs at densities ρ_a and ρ_b . To illustrate this claim, let us consider as in [14] the SSEP on a one dimensional lattice of L sites, with particles injected on site 1 and L at rates α and δ and removed from these two sites at rates γ and β . It is well known that these rates correspond to site 1 being connected to a reservoir at density $\rho_a = \frac{\alpha}{\alpha+\gamma}$ and site L to a reservoir at density $\rho_b = \frac{\delta}{\beta+\delta}$. Instead of these input rates, one could think of site 1 being connected to a large number N_1 of sites (which mimic the left reservoir), which are all initially at density ρ_a , with an exchange rate $(\alpha + \gamma)/N_1$ between site 1 and each of these N_1 sites of the reservoir. Similarly one can replace the rates β and δ by a large reservoir of N_L sites, initially at density $\rho_b = \frac{\delta}{\beta+\delta}$, with an exchange rate $(\beta + \delta)/N_L$ with site L . If initially all sites $i \leq i_0$ are at density ρ_a and all sites $i \geq i_0 + 1$ are at density ρ_b , we are in a situation where all the sites of the graph composed by the one-dimensional lattice and the reservoirs are initially occupied with probability either ρ_a or ρ_b . In this geometry the flux Q_t is then simply the total flux of particles between site i_0 and site $i_0 + 1$, and therefore, for this flux, we know from (14) that $\langle e^{\lambda Q_t} \rangle$ depends on ρ_a, ρ_b and λ through the single parameter ω .

Because the generating function $\langle e^{\lambda Q_t} \rangle$ depends only on the single parameter ω defined in (3), it can be written as

$$\langle e^{\lambda Q_t} \rangle = \sum_{n \geq 0} s_n(t) \omega^n. \quad (17)$$

Therefore, to determine the coefficients $s_n(t)$, one can limit the discussion to the particular case $\rho_b = 0$ where the analysis is easier. Then

$$\langle e^{\lambda Q_t} \rangle = \sum_{n \geq 0} s_n(t) \rho_a^n (e^\lambda - 1)^n. \quad (18)$$

This can be viewed as an expansion of $\langle e^{\lambda Q_t} \rangle$ in powers to ρ_a . We are now going to express the coefficients $s_n(t)$ in terms of properties of exclusion process on an arbitrary graph. For simplicity we consider a finite graph. Initially, only the sites of the subset A are occupied, with probability ρ_a : very much like in (11) one can write the generating function $\langle e^{\lambda Q_t} \rangle$ as

$$\langle e^{\lambda Q_t} \rangle = \sum_{E \subseteq A} \rho_a^{|E|} (1 - \rho_a)^{N_A - |E|} \sum_{q=0}^{|E|} \text{Prot}(E, q) e^{\lambda q}, \quad (19)$$

where $\text{Prot}(E, q)$ is the probability that q particles have escaped from A during t , given that at $t = 0$ only its subset E was occupied by particles. Comparing (18) and (19), we see that the coefficients $s_n(t)$ can be expressed in terms of the probabilities $\text{Prot}(E, q)$. In fact, this relation takes a simple form, which can be easily understood by taking simultaneously the limits $\rho_a \rightarrow 0$ and $\lambda \rightarrow \infty$, at fixed $\rho_a e^\lambda$, in (18) and (19): this leads to

$$s_n(t) = \sum_{E \subseteq A, |E|=n} \text{Prot}(E, n), \quad (20)$$

where the sum is over all the subsets E of n sites and $\text{Prot}(E, n)$ is the probability that, if initially all the n sites of E are occupied, the rest of the graph being empty, all the n particles have escaped from A at time t .

III. THE BETHE ANSATZ

The evolution equations of the correlation functions, on the infinite line, can be solved via the Bethe ansatz: for a fixed initial configuration, i.e. $\{\tau_x(0)\} = \{\eta_x\}$, it allows one to obtain the expression of all the correlations functions at any later time. For example the solution of (6) is

$$\langle \tau_y(t) \rangle = \sum_x P_t^{(1)}(y|x) \eta_x, \quad (21)$$

where

$$P_t^{(1)}(y|x) = \oint_{|z_1|=r} \frac{dz_1}{2i\pi z_1} z_1^{y-x} \exp \left[\left(z_1 + \frac{1}{z_1} - 2 \right) t \right] \quad (22)$$

where the integration contour is a counterclockwise circle of radius r (for convenience in what follows we choose $r \ll 1$).

Similarly one can show that the solution of (7,8) is, for $y_1 < y_2$,

$$\langle \tau_{y_1}(t) \tau_{y_2}(t) \rangle = \sum_{x_1 < x_2} P_t^{(2)}(y_1, y_2 | x_1, x_2) \eta_{x_1} \eta_{x_2}, \quad (23)$$

where

$$\begin{aligned} P_t^{(2)}(y_1, y_2 | x_1, x_2) = & \oint_{|z_1|=|z_2|=r} \frac{dz_1}{2i\pi z_1} \frac{dz_2}{2i\pi z_2} \left[z_1^{y_1-x_1} z_2^{y_2-x_2} - \left(\frac{z_1 z_2 + 1 - 2z_2}{z_1 z_2 + 1 - 2z_1} \right) z_1^{y_2-x_1} z_2^{y_1-x_2} \right] \\ & \times \exp \left[\left(z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} - 4 \right) t \right]. \end{aligned} \quad (24)$$

In fact (21) and (23) can be extended to arbitrary correlation functions to give

$$\langle \tau_{y_1}(t) \tau_{y_2}(t) \dots \tau_{y_n}(t) \rangle = \sum_{x_1 < x_2 < \dots < x_n} P_t^{(n)}(y_1, y_2, \dots, y_n | x_1, x_2, \dots, x_n) \eta_{x_1} \eta_{x_2} \dots \eta_{x_n} \quad (25)$$

where $P_t^{(n)}(y_1, y_2, \dots, y_n | x_1, x_2, \dots, x_n)$ has a Bethe ansatz expression which generalizes (22,24) (see Appendix A and [48]).

One can then use (20) to establish that the coefficients $s_n(t)$ are given by

$$s_n(t) = \sum_{x_1 < x_2 < \dots < x_n \leq 0 < y_1 < \dots < y_n} P_t^{(n)}(y_1, y_2, \dots, y_n | x_1, x_2, \dots, x_n). \quad (26)$$

For example one gets from (22, 24) for $n = 1$ and $n = 2$

$$s_1(t) = \oint_{|z_1|=r} \frac{dz_1}{2i\pi} \frac{1}{(1-z_1)^2} \exp \left[\left(z_1 + \frac{1}{z_1} - 2 \right) t \right] \quad (27)$$

and

$$\begin{aligned} s_2(t) = & \oint_{|z_1|=|z_2|=r} \frac{dz_1}{2i\pi z_1} \frac{dz_2}{2i\pi z_2} \left[\frac{z_1^2 z_2^2}{(1-z_1)(1-z_2)(1-z_1 z_2)^2} \right. \\ & \left. - \frac{z_1^3 z_2}{(1-z_1)^2 (1-z_1 z_2)^2} \left(\frac{z_1 z_2 + 1 - 2z_2}{z_1 z_2 + 1 - 2z_1} \right) \right] \times \exp \left[\left(z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} - 4 \right) t \right]. \end{aligned} \quad (28)$$

By symmetrizing the above expression over z_1 and z_2 one can show that this can be rewritten as

$$s_2(t) = \frac{1}{2} \oint_{|z_i|=r} \frac{dz_1}{2i\pi} \frac{dz_2}{2i\pi} \det \left(\frac{1}{z_k z_l + 1 - 2z_l} \right)_{1 \leq k, l \leq 2} \exp \left[\left(z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} - 4 \right) t \right]. \quad (29)$$

This expression can in fact be extended to arbitrary n , using an approach which follows closely recent works by Tracy and Widom [48, 49]. The derivation, which is detailed in Appendix A, yields

$$\langle e^{\lambda Q_t} \rangle = \sum_{n \geq 0} \omega^n s_n(t) = \sum_{n \geq 0} \frac{\omega^n}{n!} \oint_{|z_k|=r} \left[\prod_{k=1}^n \frac{dz_k e^{t(z_k+1/z_k-2)}}{2i\pi} \right] \det \left(\frac{1}{z_k z_l + 1 - 2z_l} \right)_{1 \leq k, l \leq n} \quad (30)$$

It is known [45] (see also eq.(7) of [49]) that for a Fredholm operator K which transforms a function f into a function Kf by

$$Kf(z) = \int_{|z|=r} \frac{dz}{2i\pi} K(z, z') f(z'),$$

one has

$$\det(I + \omega K) = \sum_{n \geq 0} \frac{\omega^n}{n!} \oint_{|z_k|=r} \left[\prod_{k=1}^n \frac{dz_k}{2i\pi} \right] \det(K(z_k, z_l))_{1 \leq k, l \leq n}. \quad (31)$$

This implies that

$$\log[\det(I + \omega K)] = \text{tr}[\log(I + \omega K)] = - \sum_{n \geq 0} \frac{(-\omega)^n}{n} \oint_{|z_k|=r} \left[\prod_{k=1}^n \frac{dz_k}{2i\pi} \right] K(z_1, z_2) \dots K(z_n, z_1). \quad (32)$$

Comparing (30) and (31), we see that by choosing

$$K(z, z') = \frac{\exp \left[\left(z + \frac{1}{z} - 2 \right) t \right]}{zz' + 1 - 2z}, \quad (33)$$

one gets (32) that

$$\log \langle e^{\lambda Q_t} \rangle = - \sum_{n \geq 1} \frac{(-\omega)^n}{n} \oint_{|z_k|=r} \left[\prod_{k=1}^n \frac{dz_k}{2i\pi} \right] \prod_{k=1}^n \left(\frac{\exp \left[\left(z_k + \frac{1}{z_k} - 2 \right) t \right]}{z_k z_{k+1} + 1 - 2z_{k+1}} \right) \quad (34)$$

(with the convention that $z_{n+1} \equiv z_1$ in the n -th term).

In Appendix B, we derive the large t behavior of the integrals in the r.h.s. of (34), which leads to

$$\log \langle e^{\lambda Q_t} \rangle \sim - \sum_{n \geq 1} \frac{(-\omega)^n}{n} \sqrt{\frac{t}{\pi n}} = \sqrt{t} F(\omega),$$

completing the derivation of (1,2).

Remark : from the expressions (21, 22, 23, 24, 25, A1), one can in principle calculate arbitrary correlation functions of the occupation numbers τ_i at time t . For the one-point and the connected two-point functions one gets when $\rho_a = 1$ and $\rho_b = 0$

$$\langle \tau_x \rangle = \oint \frac{dz e^{t(z+1/z-2)}}{2i\pi z} \frac{z^x}{1-z} \quad \text{and} \quad \langle \tau_x \tau_y \rangle_c = \oint \frac{dz dz' e^{t(z+1/z+z'+1/z'-4)}}{4\pi^2 z z'} \frac{z^x z'^y}{zz' + 1 - 2z'}.$$

Then, carrying out an asymptotic analysis similar to the one performed in Appendix B yields for $n = 1$ and $n = 2$

$$\langle \tau_{x_1} \dots \tau_{x_n} \rangle_c \simeq t^{\frac{1-n}{2}} G_n \left(\frac{x_1}{\sqrt{t}}, \dots, \frac{x_n}{\sqrt{t}} \right). \quad (35)$$

with

$$G_1(X) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{X/2} e^{-u^2} du \quad \text{and} \quad G_2(X, Y) = -\frac{e^{-(X+Y)^2/8}}{2\sqrt{2}\pi} \int_{-\infty}^{\frac{X-Y}{2\sqrt{2}}} e^{-u^2} du. \quad (36)$$

The scaling form (35,36) of the two-point function is very reminiscent of what is known for the steady state of the SSEP on an open interval [46] of length L , where the two point function scales like L^{-1} and the n -point (connected) correlation function scales like L^{1-n} [18]. It would be interesting to know whether the scaling form (35) remains valid for $n \geq 2$. If so one could then try to determine the scaling functions G_n in order to obtain the large deviation function of an arbitrary density profile [6, 16].

IV. CONCLUSION

In the present work, we have shown that, for a step initial density profile, the generating function of the integrated current Q_t is a function of a single parameter ω , defined in (3), which takes a simple closed expression (1). This ω dependence is also valid

for an arbitrary graph. In one dimension, for this non-equilibrium initial condition, the distribution of Q_t is clearly not Gaussian with a tail which decays faster than a Gaussian (5).

It would be interesting to generalize our results to other diffusive systems and to try to calculate the distribution of Q_t with the same step initial condition, in order to see under which conditions one could recover the non-Gaussian decay (5). The most promising approach, at the moment, is to try to generalize the macroscopic theory of Bertini, De Sole, Gabrielli, Jona-Lasinio, and Landim [6, 7, 8] to a non-steady state initial condition.

A possible extension of the present work would be to introduce a weak asymmetry (of order $t^{-1/2}$) in the hopping rates to understand the cross-over in the current fluctuations between the SSEP and the ASEP.

Another interesting question would be to determine the scaling form (35) of all the higher correlation functions and to obtain the large deviation function of an arbitrary density profile, when the initial condition is, as in figure 1 far from a steady state situation.

APPENDIX A: DETERMINANT EXPRESSION (30) OF $\langle e^{\lambda Q_t} \rangle$

In this appendix we derive the general expression (30) of the (s_n) from (26). Our derivation relies heavily on results obtained by Tracy and Widom on the particle trajectories of an ASEP model with the same geometry[48, 49]. In the present appendix we will use the Bethe ansatz itself ([48], Theorem 2.1),

$$P_t^{(n)}(y_1, y_2, \dots, y_n | x_1, x_2, \dots, x_n) = \sum_{\sigma} \text{sgn}(\sigma) \oint \left[\prod_{k=1}^n \frac{dz_k}{2i\pi z_k} e^{t(z_k+1/z_k-2)} z_{\sigma(k)}^{y_k-x_{\sigma(k)}} \right] \left[\prod_{k < l} \frac{z_{\sigma(k)} z_{\sigma(l)} + 1 - 2z_{\sigma(k)}}{z_k z_l + 1 - 2z_k} \right], \quad (\text{A1})$$

as well as two algebraic identities ([48], eq. (1.6) and [49], eq. (7)),

$$\sum_{\sigma} \text{sgn}(\sigma) \left[\prod_{k=1}^n \frac{z_{\sigma(k)} \dots z_{\sigma(n)}}{1 - z_{\sigma(k)} \dots z_{\sigma(n)}} \right] \left[\prod_{k < l} (z_{\sigma(k)} z_{\sigma(l)} + 1 - 2z_{\sigma(k)}) \right] = z_1 \dots z_n \frac{\prod_{k < l} (z_l - z_k)}{\prod_k (1 - z_k)} \quad (\text{A2})$$

$$\left[\prod_{k=1}^n \frac{1}{(1 - z_k)^2} \right] \left[\prod_{k \neq l} \frac{z_k - z_l}{z_k z_l + 1 - 2z_k} \right] = \det \left(\frac{1}{z_k z_l + 1 - 2z_k} \right)_{1 \leq k, l \leq n}. \quad (\text{A3})$$

In order to obtain s_n from (26), one has to sum $P_t^{(n)}(y_1, y_2, \dots, y_n | x_1, x_2, \dots, x_n)$ over $x_1 < \dots < x_n \leq 0$ and $0 < y_1 < \dots < y_n$: this yields

$$\sum_{x_1 < \dots < x_n \leq 0} \prod_k z_k^{-x_k} = \frac{1}{z_1 \dots z_n} \left[\prod_{k=1}^n \frac{z_1 \dots z_k}{1 - z_1 \dots z_k} \right] \text{ and } \sum_{0 < y_1 < \dots < y_n} \prod_k z_{\sigma(k)}^{y_k} = \prod_{k=1}^n \frac{z_{\sigma(k)} \dots z_{\sigma(n)}}{1 - z_{\sigma(k)} \dots z_{\sigma(n)}}.$$

Therefore

$$s_n = \sum_{\sigma} \text{sgn}(\sigma) \oint \left[\prod_{k=1}^n \frac{dz_k e^{t(z_k+1/z_k-2)}}{2i\pi z_k^2} \frac{z_1 \dots z_k}{1 - z_1 \dots z_k} \frac{z_{\sigma(k)} \dots z_{\sigma(n)}}{1 - z_{\sigma(k)} \dots z_{\sigma(n)}} \right] \left[\prod_{k < l} \frac{z_{\sigma(k)} z_{\sigma(l)} + 1 - 2z_{\sigma(k)}}{z_k z_l + 1 - 2z_k} \right].$$

We are now in position to use (A2), which yields

$$s_n = \oint \left[\prod_{k=1}^n \frac{dz_k e^{t(z_k+1/z_k-2)}}{2i\pi z_k (1 - z_k)} \frac{z_1 \dots z_k}{1 - z_1 \dots z_k} \right] \left[\prod_{k < l} \frac{z_l - z_k}{z_k z_l + 1 - 2z_k} \right].$$

In order to eliminate the remaining $\frac{z_1 \dots z_k}{1 - z_1 \dots z_k}$, it is necessary to use (A2) again. As the factors in (A2) are of the form $\frac{z_k \dots z_n}{1 - z_k \dots z_n}$, we first relabel $z_k \rightarrow z_{n-k}$, so that

$$\begin{aligned} s_n &= \oint \left[\prod_{k=1}^n \frac{dz_k e^{t(z_k+1/z_k-2)}}{2i\pi z_k (1 - z_k)} \frac{z_k \dots z_n}{1 - z_k \dots z_n} \right] \left[\prod_{k > l} \frac{z_l - z_k}{z_k z_l + 1 - 2z_k} \right] \\ &= \oint \left[\prod_{k=1}^n \frac{dz_k e^{t(z_k+1/z_k-2)}}{2i\pi z_k (1 - z_k)} \right] \frac{\prod_{k < l} (z_k - z_l)}{\prod_{k \neq l} (z_k z_l + 1 - 2z_k)} \left[\prod_{k < l} (z_k z_l + 1 - 2z_k) \right] \left[\prod_{k=1}^n \frac{z_k \dots z_n}{1 - z_k \dots z_n} \right]. \end{aligned}$$

We then replace the expression above by its average over all permutations of the (z_k) in order to apply (A2). The first term of the product is unchanged and the second term gets a sign factor $\text{sgn}(\sigma)$. Hence

$$\begin{aligned} s_n &= \sum_{\sigma} \frac{\text{sgn}(\sigma)}{n!} \oint \left[\prod_{k=1}^n \frac{dz_k e^{t(z_k+1/z_k-2)}}{2i\pi z_k(1-z_k)} \right] \left[\prod_{k < l} (z_k - z_l) \right] \frac{\prod_{k < l} (z_{\sigma(k)} z_{\sigma(l)} + 1 - 2z_{\sigma(k)})}{\prod_{k \neq l} (z_k z_l + 1 - 2z_k)} \left[\prod_{k=1}^n \frac{z_{\sigma(k)} \dots z_{\sigma(n)}}{1 - z_{\sigma(k)} \dots z_{\sigma(n)}} \right] \\ &= \frac{1}{n!} \oint \left[\prod_{k=1}^n \frac{dz_k e^{t(z_k+1/z_k-2)}}{2i\pi(1-z_k)^2} \right] \left[\prod_{k \neq l} \frac{z_k - z_l}{z_k z_l + 1 - 2z_k} \right]. \end{aligned}$$

Applying (A3) then leads to

$$s_n(t) = \frac{1}{n!} \oint \left[\prod_{k=1}^n \frac{dz_k}{2i\pi} \right] \det \left(\frac{e^{t(z_k+1/z_k-2)}}{z_k z_l + 1 - 2z_k} \right)_{1 \leq k, l \leq n},$$

which is the expression (30).

APPENDIX B: DERIVATION OF THE ASYMPTOTICS (34) OF THE GENERATING FUNCTION

In this appendix we derive the large t behavior of the integrals $I_n = \text{tr } K^n$, which appear in (32,33,34). I_n can be expressed as

$$I_n = \text{tr } K_t^n = \oint \frac{dz_1}{2i\pi} \dots \frac{dz_n}{2i\pi} K_t(z_1, z_2) \dots K_t(z_n, z_1) = \oint \prod_{k=1}^n \frac{dz_k}{2i\pi} \frac{e^{t(z_k+1/z_k-2)}}{z_k z_{k+1} + 1 - 2z_k},$$

where the integration contour of each z_k is a circle of radius $r \ll 1$ (and by convention we have $z_{n+k} \equiv z_k$). By using the identity

$$\frac{e^{t(z_{k+1}+1/z_k-2)}}{z_{k+1} + 1/z_k - 2} = \frac{1}{z_{k+1} + 1/z_k - 2} + \int_0^t dt_k e^{t_k(z_{k+1}+1/z_k-2)},$$

I_n can be expressed as

$$I_n = \sum_{E \subset \{1, \dots, n\}} \oint \left[\prod_{k=1}^n \frac{dz_k}{2i\pi z_k} \right] \left[\prod_{k \notin E} \frac{1}{z_{k+1} + 1/z_k - 2} \right] \left[\prod_{k \in E} \int_0^t dt_k e^{t_k(z_{k+1}+1/z_k-2)} \right].$$

If $E \neq \{1, \dots, n\}$, then there exists at least one k such that $k \notin E$. For fixed values of the $(z_l)_{l \neq k}$ and the $(t_l)_{l \neq k}$, the integral over z_k can be written either as

$$\oint \frac{dz_k}{2i\pi z_k} \frac{e^{t_{k-1} z_k}}{z_{k+1} + 1/z_k - 2} \quad \text{or as} \quad \oint \frac{dz_k}{2i\pi z_k} \frac{1}{(z_{k+1} + 1/z_k - 2)(z_k + 1/z_{k-1} - 2)}.$$

As the integration contour is a circle of very small radius these expressions obviously vanish. Therefore the case $E = \{1, \dots, n\}$ gives the only non-zero contribution to I_n :

$$I_n = \oint_{|z_k|=r} \int_0^t \prod_{k=1}^n \frac{dz_k dt_k}{2i\pi z_k} e^{t_k(z_{k+1}+1/z_k-2)}.$$

The integrals over the z_k 's can now be evaluated using the saddle-point method around $z_k = \sqrt{t_k/t_{k-1}}$. This yields

$$I_n \simeq \int_0^t \prod_{k=1}^n \frac{dt_k}{2\sqrt{\pi t_k}} e^{-(\sqrt{t_k} - \sqrt{t_{k-1}})^2} = \left(\frac{t}{\pi} \right)^{n/2} \int_0^1 \prod d\alpha_k e^{-t(\alpha_k - \alpha_{k-1})^2}.$$

Finally, since the integrand above is only non-vanishing when $|\alpha_k - \alpha_l| \sim 1/\sqrt{t}$ for $t \rightarrow \infty$, one gets

$$I_n \simeq \left(\frac{t}{\pi} \right)^{n/2} \int_0^1 d\alpha_1 \int_{-\infty}^{\infty} d\alpha_2 \dots d\alpha_n e^{-t \sum (\alpha_k - \alpha_{k-1})^2} = \sqrt{\frac{t}{\pi n}}.$$

and this completes the proof of (34).

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